

# NEW CONGRUENCES OF PARTITIONS WITH ODD PARTS DISTINCT

LIUQUAN WANG

ABSTRACT. Let  $\text{pod}(n)$  denote the number of partitions of  $n$  with odd parts distinct, and  $r_k(n)$  be the number of representations of  $n$  as sum of  $k$  squares. We find the following two arithmetic relations: for any integer  $n \geq 0$ ,

$$\text{pod}(3n+2) \equiv 2(-1)^{n+1}r_5(8n+5) \pmod{9},$$

and

$$\text{pod}(5n+2) \equiv 2(-1)^n r_3(8n+3) \pmod{5}.$$

From which we deduce many interesting congruences including the following two infinite families of Ramanujan-type congruences: for  $a \in \{11, 19\}$  and any integers  $\alpha \geq 1$  and  $n \geq 0$ , we have

$$\text{pod}\left(5^{2\alpha+2}n + \frac{a \cdot 5^{2\alpha+1} + 1}{8}\right) \equiv 0 \pmod{5}.$$

## 1. INTRODUCTION

Let  $\psi(q)$  be one of Ramanujan's theta functions, namely

$$\psi(q) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2; q^2)_{\infty}^2}{(q; q)_{\infty}}.$$

We denote by  $\text{pod}(n)$  the number of partitions of  $n$  with odd parts distinct. The generating function of  $\text{pod}(n)$  is

$$\sum_{n=0}^{\infty} \text{pod}(n)q^n = \frac{(-q; q^2)_{\infty}}{(q^2; q^2)_{\infty}} = \frac{1}{\psi(-q)}. \quad (1)$$

The arithmetic properties of  $\text{pod}(n)$  were first studied by Hirschhorn and Sellers [4] in 2010. They obtained some interesting congruences involving the following infinite family of Ramanujan-type congruences: for any integers  $\alpha \geq 0$  and  $n \geq 0$ ,

$$\text{pod}\left(3^{2\alpha+3}n + \frac{23 \times 3^{2\alpha+2} + 1}{8}\right) \equiv 0 \pmod{3}.$$

Later on Radu and Sellers [7] obtained other deep congruences for  $\text{pod}(n)$  modulo 5 and 7, such as

$$\text{pod}(135n+8) \equiv \text{pod}(135n+107) \equiv \text{pod}(135n+116) \equiv 0 \pmod{5}, \quad \text{and}$$

$$\text{pod}(567n+260) \equiv \text{pod}(567n+449) \equiv 0 \pmod{7}.$$

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For nonnegative integers  $n$  and  $k$ , let  $r_k(n)$  (resp.  $t_k(n)$ ) denote the number of representations of  $n$  as sum of  $k$  squares (resp. triangular numbers). In 2011, based on the generating function of  $\text{pod}(3n+2)$  found in [4], Lovejoy and Osburn discovered the following arithmetic relation:

$$\text{pod}(3n+2) \equiv (-1)^n r_5(8n+5) \pmod{3}. \quad (2)$$

Following their steps, we will present some new congruences modulo 5 and 9 for  $\text{pod}(n)$ .

Firstly, we find that (2) can be improved to a congruence modulo 9.

**Theorem 1.** *For any integer  $n \geq 0$ , we have*

$$\text{pod}(3n+2) \equiv 2(-1)^{n+1} r_5(8n+5) \pmod{9}.$$

The following result will be a consequence of Theorem 1 upon invoking some properties of  $r_5(n)$ .

**Theorem 2.** *Let  $p \geq 3$  be a prime, and  $N$  be a positive integer such that  $pN \equiv 5 \pmod{8}$ . Let  $\alpha$  be any nonnegative integer.*

(1) *If  $p \equiv 1 \pmod{3}$ , then*

$$\text{pod}\left(\frac{3p^{6\alpha+5}N+1}{8}\right) \equiv 0 \pmod{3},$$

and

$$\text{pod}\left(\frac{3p^{18\alpha+17}N+1}{8}\right) \equiv 0 \pmod{9}.$$

(2) *If  $p \equiv 2 \pmod{3}$ , then*

$$\text{pod}\left(\frac{3p^{4\alpha+3}N+1}{8}\right) \equiv 0 \pmod{9}.$$

Secondly, with the same method used in proving Theorem 1, we can establish a similar congruence for  $\text{pod}(n)$  modulo 5.

**Theorem 3.** *For any integer  $n \geq 0$ , we have*

$$\text{pod}(5n+2) \equiv 2(-1)^n r_3(8n+3) \pmod{5}.$$

Some miscellaneous congruences can be deduced from this theorem.

**Theorem 4.** *For any integers  $n \geq 0$  and  $\alpha \geq 1$ , we have*

$$\text{pod}\left(5^{2\alpha+2}n + \frac{11 \cdot 5^{2\alpha+1} + 1}{8}\right) \equiv 0 \pmod{5},$$

and

$$\text{pod}\left(5^{2\alpha+2}n + \frac{19 \cdot 5^{2\alpha+1} + 1}{8}\right) \equiv 0 \pmod{5}.$$

**Theorem 5.** *Let  $p \equiv 4 \pmod{5}$  be a prime, and  $N$  be a positive integer which is coprime to  $p$  such that  $pN \equiv 3 \pmod{8}$ . We have*

$$\text{pod}\left(\frac{5p^3N+1}{8}\right) \equiv 0 \pmod{5}.$$

For example, let  $p = 19$  and  $N = 8n+1$  where  $n \geq 0$  and  $n \not\equiv 7 \pmod{19}$ , we have

$$\text{pod}(34295n + 4287) \equiv 0 \pmod{5}.$$

**Theorem 6.** *Let  $p \geq 3$  be a prime, and  $N$  be a positive integer which is not divisible by  $p$  such that  $pN \equiv 3 \pmod{8}$ . Let  $\alpha$  be any nonnegative integer.*

*(1) If  $p \equiv 1 \pmod{5}$ , we have*

$$\text{pod}\left(\frac{5p^{10\alpha+9}N+1}{8}\right) \equiv 0 \pmod{5}.$$

*(2) If  $p \equiv 2, 3, 4 \pmod{5}$ , we have*

$$\text{pod}\left(\frac{5p^{8\alpha+7}N+1}{8}\right) \equiv 0 \pmod{5}.$$

## 2. PRELIMINARIES

**Lemma 1.** *(Cf. [7, Lemma 1.2].) Let  $p$  be a prime and  $\alpha$  be a positive integer. Then*

$$(q; q)_{\infty}^{p^{\alpha}} \equiv (q^p; q^p)_{\infty}^{p^{\alpha-1}} \pmod{p^{\alpha}}.$$

**Lemma 2.** *For any prime  $p \geq 3$ , we have*

$$t_4\left(pn + \frac{p-1}{2}\right) \equiv t_4(n) \pmod{p}, \quad t_8(pn + p - 1) \equiv t_8(n) \pmod{p^3}.$$

*Proof.* By [2, Theorem 3.6.3], we know  $t_4(n) = \sigma(2n+1)$ . For any positive integer  $N$ , we have

$$\sigma(N) = \sum_{d|N, p|d} d + \sum_{d|N, p \nmid d} d \equiv \sum_{d|N, p \nmid d} d \pmod{p}.$$

Let  $N = 2n+1$  and  $N = p(2n+1)$  respectively. It is easy to deduce that  $\sigma(p(2n+1)) \equiv \sigma(2n+1) \pmod{p}$ . This clearly implies the first congruence.

From [2, equation (3.8.3) in page 81], we know

$$t_8(n) = \sum_{\substack{d|(n+1) \\ d \text{ odd}}} \left(\frac{n+1}{d}\right)^3.$$

By a similar argument we can prove the second congruence. □

**Lemma 3.** *(Cf. [1].) For  $1 \leq k \leq 7$ , we have*

$$r_k(8n+k) = 2^k \left(1 + \frac{1}{2} \binom{k}{4}\right) t_k(n).$$

**Lemma 4.** *(Cf. [3].) Let  $p \geq 3$  be a prime and  $n$  be a positive integer such that  $p^2 \nmid n$ . For any integer  $\alpha \geq 0$ , we have*

$$r_5(p^{2\alpha}n) = \left(\frac{p^{3\alpha+3}-1}{p^3-1} - p\left(\frac{n}{p}\right)\frac{p^{3\alpha}-1}{p^3-1}\right) r_5(n),$$

where  $\left(\frac{\cdot}{p}\right)$  denotes the Legendre symbol.

**Lemma 5.** *(Cf. [5].) Let  $p \geq 3$  be a prime. For any integers  $n \geq 1$  and  $\alpha \geq 0$ , we have*

$$r_3(p^{2\alpha}n) = \left(\frac{p^{\alpha+1}-1}{p-1} - \left(\frac{-n}{p}\right)\frac{p^{\alpha}-1}{p-1}\right) r_3(n) - p\frac{p^{\alpha}-1}{p-1} r_3(n/p^2).$$

Here we take  $r_3(n/p^2) = 0$  unless  $p^2|n$ .

## 3. PROOFS OF THE THEOREMS

*Proof of Theorem 1.* Let  $p = 3$  in Lemma 2. We deduce that  $t_8(3n + 2) \equiv t_8(n) \pmod{9}$ . By (1) we have

$$\psi(q)^9 \sum_{n=0}^{\infty} \text{pod}(n)(-q)^n = \psi(q)^8 = \sum_{n=0}^{\infty} t_8(n)q^n.$$

By Lemma 1 we obtain  $\psi(q)^9 \equiv \psi(q^3)^3 \pmod{9}$ . Hence

$$\psi(q^3)^3 \sum_{n=0}^{\infty} \text{pod}(n)(-q)^n \equiv \sum_{n=0}^{\infty} t_8(n)q^n \pmod{9}.$$

If we extract those terms of the form  $q^{3n+2}$  on both sides, we obtain

$$\psi(q^3)^3 \sum_{n=0}^{\infty} \text{pod}(3n+2)(-q)^{3n+2} \equiv \sum_{n=0}^{\infty} t_8(3n+2)q^{3n+2} \pmod{9}.$$

Dividing both sides by  $q^2$ , then replacing  $q^3$  by  $q$ , we get

$$\psi(q)^3 \sum_{n=0}^{\infty} \text{pod}(3n+2)(-q)^n \equiv \sum_{n=0}^{\infty} t_8(3n+2)q^n \equiv \sum_{n=0}^{\infty} t_8(n)q^n = \psi(q)^8 \pmod{9}.$$

Hence

$$\sum_{n=0}^{\infty} \text{pod}(3n+2)(-q)^n \equiv \psi(q)^5 \equiv \sum_{n=0}^{\infty} t_5(n)q^n \pmod{9}.$$

Comparing the coefficients of  $q^n$  on both sides, we deduce that  $\text{pod}(3n+2) \equiv (-1)^n t_5(n) \pmod{9}$ .

Let  $k = 5$  in Lemma 3. We obtain  $t_5(n) = r_5(8n+5)/112$ , and from this the theorem follows.  $\square$

*Proof of Theorem 2.* (1) Let  $n = pN$  in Lemma 4, and then we replace  $\alpha$  by  $3\alpha+2$ . Since

$$\frac{p^{9\alpha+9} - 1}{p^3 - 1} = 1 + p^3 + \cdots + p^{3(3\alpha+2)} \equiv 0 \pmod{3},$$

we deduce that  $r_5(p^{6\alpha+5}N) \equiv 0 \pmod{3}$ .

Let  $n = \frac{p^{6\alpha+5}N-5}{8}$  in Theorem 1. We deduce that  $\text{pod}(\frac{3p^{6\alpha+5}N+1}{8}) \equiv 0 \pmod{3}$ .

Similarly, let  $n = pN$  in Lemma 4 and we replace  $\alpha$  by  $9\alpha+8$ . Since  $p \equiv 1 \pmod{3}$  implies  $p^3 \equiv 1 \pmod{9}$ , we have

$$\frac{p^{27\alpha+27} - 1}{p^3 - 1} = 1 + p^3 + \cdots + p^{3(9\alpha+8)} \equiv 0 \pmod{9}.$$

Hence  $r_5(p^{18\alpha+17}N) \equiv 0 \pmod{9}$ .

Let  $n = \frac{p^{18\alpha+17}N-5}{8}$  in Theorem 1. We deduce that  $\text{pod}(\frac{3p^{18\alpha+17}N+1}{8}) \equiv 0 \pmod{9}$ .

(2) Let  $n = pN$  in Lemma 4, and then we replace  $\alpha$  by  $2\alpha+1$ . Note that  $p \equiv 2 \pmod{3}$  implies  $p^3 \equiv -1 \pmod{9}$ . Since  $p^{6\alpha+6} \equiv 1 \pmod{9}$ , we have  $r_5(p^{4\alpha+3}N) \equiv 0 \pmod{9}$ .

Let  $n = \frac{p^{4\alpha+3}N-5}{8}$  in Theorem 1. We complete our proof.  $\square$

*Proof of Theorem 3.* Let  $p = 5$  in Lemma 2. We deduce that  $t_4(5n + 2) \equiv t_4(n) \pmod{5}$ .

By (1) we have

$$\psi(q)^5 \sum_{n=0}^{\infty} \text{pod}(n)(-q)^n = \psi(q)^4 = \sum_{n=0}^{\infty} t_4(n)q^n.$$

By Lemma 1 we obtain  $\psi(q)^5 \equiv \psi(q^5) \pmod{5}$ . Hence

$$\psi(q^5) \sum_{n=0}^{\infty} \text{pod}(n)(-q)^n \equiv \sum_{n=0}^{\infty} t_4(n)q^n \pmod{5}.$$

If we extract those terms of the form  $q^{5n+2}$  on both sides, we obtain

$$\psi(q^5) \sum_{n=0}^{\infty} \text{pod}(5n+2)(-q)^{5n+2} \equiv \sum_{n=0}^{\infty} t_4(5n+2)q^{5n+2} \pmod{5}.$$

Dividing both sides by  $q^2$ , and then replacing  $q^5$  by  $q$ , we get

$$\psi(q) \sum_{n=-\infty}^{\infty} \text{pod}(5n+2)(-q)^n \equiv \sum_{n=0}^{\infty} t_4(5n+2)q^n \equiv \sum_{n=0}^{\infty} t_4(n)q^n = \psi(q)^4 \pmod{5}.$$

Hence we have

$$\sum_{n=0}^{\infty} \text{pod}(5n+2)(-q)^n \equiv \psi(q)^3 = \sum_{n=0}^{\infty} t_3(n)q^n \pmod{5}.$$

Comparing the coefficients of  $q^n$  on both sides, we deduce that  $\text{pod}(5n+2) \equiv (-1)^n t_3(n) \pmod{5}$ .

Let  $k = 3$  in Lemma 3. We obtain  $t_3(n) = r_3(8n+3)/8$ , from which the theorem follows.  $\square$

*Proof of Theorem 4.* Let  $p = 5$  and  $n = 5m + r$  ( $r \in \{1, 4\}$ ) in Lemma 5. Since  $(\frac{-r}{5}) = 1$ , we deduce that  $r_3(5^{2\alpha}(5m+r)) \equiv 0 \pmod{5}$  for any integer  $\alpha \geq 1$ .

Let  $n = \frac{5^{2\alpha}(40m+a)-3}{8}$  ( $a \in \{11, 19\}$ ). By Theorem 3, we have

$$r_3(8n+3) = r_3(5^{2\alpha}(40m+a)) \equiv 0 \pmod{5}.$$

Hence

$$\text{pod}\left(5^{2\alpha+2}m + \frac{a \cdot 5^{2\alpha+1} + 1}{8}\right) = \text{pod}(5n+2) \equiv 2(-1)^n r_3(8n+3) \equiv 0 \pmod{5}.$$

$\square$

*Proof of Theorem 5.* Let  $\alpha = 1$  and  $n = pN$  in Lemma 5. We have

$$r_3(p^3N) = (1+p)r_3(pN) \equiv 0 \pmod{5}.$$

Let  $n = \frac{p^3N-3}{8}$  in Theorem 3, we have

$$\text{pod}\left(\frac{5p^3N+1}{8}\right) = \text{pod}(5n+2) \equiv 2(-1)^n r_3(8n+3) = 2(-1)^n r_3(p^3N) \equiv 0 \pmod{5}.$$

$\square$

*Proof of Theorem 6.* (1) Let  $n = pN$  in Lemma 5, and then we replace  $\alpha$  by  $5\alpha + 4$ . We have

$$\frac{p^{5\alpha+5} - 1}{p - 1} = 1 + p + \cdots + p^{5\alpha+4} \equiv 0 \pmod{5}.$$

Hence  $r_3(p^{10\alpha+9}N) \equiv 0 \pmod{5}$ . Let  $n = \frac{p^{10\alpha+9}N-3}{8}$  in Theorem 3. We have

$$\text{pod}\left(\frac{5p^{10\alpha+9}N + 1}{8}\right) = \text{pod}(5n + 2) \equiv 2(-1)^n r_3(p^{10\alpha+9}N) \equiv 0 \pmod{5}.$$

(2) Let  $n = pN$  in Lemma 5, and then we replace  $\alpha$  by  $4\alpha + 3$ . Since  $p^{4\alpha+4} \equiv 1 \pmod{5}$ , we deduce that  $r_3(p^{8\alpha+7}N) \equiv 0 \pmod{5}$ . Let  $n = \frac{p^{8\alpha+7}N-3}{8}$  in Theorem 3, we have

$$\text{pod}\left(\frac{5p^{8\alpha+7}N + 1}{8}\right) = \text{pod}(5n + 2) \equiv 2(-1)^n r_3(p^{8\alpha+7}N) \equiv 0 \pmod{5}.$$

□

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DEPARTMENT OF MATHEMATICS, NATIONAL UNIVERSITY OF SINGAPORE, SINGAPORE, 119076  
*E-mail address:* wangliuquan@u.nus.edu; mathlwang@163.com